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Third-order ODEs and four-dimensional split signature Einstein metrics

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Abstract

We construct a family of split signature Einstein metrics in four dimensions, corresponding to particular classes of third-order ODEs considered modulo fiber preserving transformations of variables.

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Keywords: Einstein metrics; Third-order ODE; Transformations of variables

1. Introduction

Our starting point is a third-order ordinary differential equation (ODE)

$$y''' = F(x, y, y', y''), \tag{1}$$

for a real function $y = y(x)$. Here $F = F(x, y, p, q)$ is a sufficiently smooth real function of four real variables $(x, y, p = y', q = y'')$.

Given another third-order ODE

$$\bar{y}''' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'') \tag{2}$$

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it is often convenient to know whether there exists a suitable transformation of variables $(x, y, p, q) \to (\bar{x}, \bar{y}, \bar{p}, \bar{q})$ which brings (2) to (1). Several types of such transformations are of particular importance. Here we consider fiber preserving (f.p.) transformations, which are of the form

$$\bar{x} = \bar{x}(x), \qquad \bar{y} = \bar{y}(x, y). \tag{3}$$

We say that two third-order ODEs, (1) and (2), are (locally) f.p. equivalent iff there exists a (local) f.p. transformation (3), which brings (2) to (1). The task of finding necessary and sufficient conditions for ODEs (1) and (2) to be (locally) f.p. equivalent, is called a f.p. equivalence problem for third-order ODEs. In the cases of (more general) point transformations and contact transformations, this problem was studied and solved by Cartan [1] and Chern [2] in the years 1939–1941. The interest in these studies has been recently revived due to the fact that important equivalence classes of third-order ODEs naturally define three-dimensional conformal Lorentzian structures including Einstein–Weyl structures. This makes these equivalence problems aplicable not only to differential geometry but also to the theory of integrable systems and general relativity [3,8,11].

In this paper we show how to construct four-dimensional split signature Einstein metrics, starting from particular ODEs of third-order. We formulate the problem of f.p. equivalence in terms of differential forms. Invoking Cartan's equivalence method, we construct a six-dimensional manifold with a distinguished coframe on it, which encodes all information about original equivalence problem. For specific types of the ODEs, the class of Einstein metrics can be explicitly constructed from this coframe. This result is a byproduct of the full solution of the f.p. equivalence problem, that will be described in [5].

We acknowledge that all our calculations were checked by the independent use of the two symbolic calculations programs: Maple and Mathematica.

2. Third-order ODE and Cartan's method

Following Cartan and Chern, we rewrite (1), using 1-forms

$$\begin{aligned}
\omega^1 &= \mathrm{d}y - p\,\mathrm{d}x,\\
\omega^2 &= \mathrm{d}p - q\,\mathrm{d}x,\\
\omega^3 &= \mathrm{d}q - F(x, y, p, q)\,\mathrm{d}x,\\
\omega^4 &= \mathrm{d}x.
\end{aligned} \tag{4}$$

These are defined on the second jet space $\mathcal{J}^2$ locally parametrized by (x, y, p, q). Each solution $y = f(x)$ of (1) is fully described by the two conditions: forms $\omega^1, \omega^2, \omega^3$ vanish on a curve $(t, f(t), f'(t), f''(t))$ and, as this defines a solution up to transformations of x, $\omega^4 = \mathrm{d}t$ on this curve. Suppose now, that Eq. (1) undergoes fiber preserving transformations

(3). Then the forms (4) transform by

$$\begin{aligned}
&\omega^1 \to \bar{\omega}^1 = \alpha\omega^1,\\
&\omega^2 \to \bar{\omega}^2 = \beta(\omega^2 + \gamma\omega^1),\\
&\omega^3 \to \bar{\omega}^3 = \epsilon(\omega^3 + \eta\omega^2 + \varkappa\omega^1),\\
&\omega^4 \to \bar{\omega}^4 = \lambda\omega^4,
\end{aligned} \tag{5}$$

where functions $\alpha, \beta, \gamma, \epsilon, \eta, \varkappa, \lambda$ are defined on $\mathcal{J}^2$, satisfy $\alpha\beta\epsilon\lambda \neq 0$ and are determined by a particular choice of transformation (3). A fiber preserving equivalence class of ODEs is described by forms (4) defined up to transformations (5). Eqs. (1) and (2) are f.p. equivalent, iff their corresponding forms (ω^i) and $(\bar{\omega}^j)$ are related as above.

We now apply Cartan's equivalence method [9,10]. Its key idea is to enlarge the space $\mathcal{J}^2$ to a new manifold $\tilde{\mathcal{P}}$, on which functions $\alpha, \beta, \gamma, \epsilon, \eta, \varkappa, \lambda$ are additional coordinates. The coframe (ω^i) defined up to transformations (5), is now replaced by a set of four well-defined 1-forms

$$\begin{aligned}
&\theta^1 = \alpha\omega^1,\\
&\theta^2 = \beta(\omega^2 + \gamma\omega^1),\\
&\theta^3 = \epsilon(\omega^3 + \eta\omega^2 + \varkappa\omega^1),\\
&\theta^4 = \lambda\omega^4
\end{aligned}$$

on $\tilde{\mathcal{P}}$. If, in addition, the following f.p. invariant condition [4,6]

$$F_{qq} \neq 0$$

is satisfied then, there is a geometrically distinguished way of choosing five parameters $\beta, \epsilon, \eta, \varkappa, \lambda$ to be functions of $(x, y, p, q, \alpha, \gamma)$. Then, on a six-dimensional manifold $\mathcal{P}$ parametrized by $(x, y, p, q, \alpha, \gamma)$ Cartan's method gives a way of supplementing the well-defined four 1-forms (θ^i) with two other 1-forms Ω^1, Ω^2 so that the set $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2)$ constitutes a rigid coframe on $\mathcal{P}$. According to the theory of G-structures [7,10], all information on a f.p. equivalence class of Eq. (1) satisfying $F_{qq} \neq 0$ is encoded in the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2)$. Two Eqs. (1) and (2) are f.p. equivalent, iff there exists a diffeomorphism $\psi : \mathcal{P} \to \bar{\mathcal{P}}$, such that $\psi^*\bar{\theta}^i = \theta^i$, $\psi^*\bar{\Omega}^A = \Omega^A$, where $i = 1, 2, 3, 4$ and $A = 1, 2$. The procedure of constructing manifold $\mathcal{P}$ and the coframe (θ^i, Ω^A) is explained in details in [9,10] for a general case and in [4,5] for this specific problem. Here we omit the details of this procedure, summarizing the results on f.p. equivalence problem in the following theorem.

Theorem 2.1. *A third-order ODE $y''' = F(x, y, y', y'')$, satisfying $F_{qq} \neq 0$, considered modulo fiber preserving transformations of variables, uniquely defines a six-dimensional manifold $\mathcal{P}$, and an invariant coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2)$ on it. In local coordinates*

$(x, y, p = y', q = y'', \alpha, \gamma)$ *this coframe is given by*

$$\begin{aligned}
\theta^1 &= \alpha\omega^1,\\
\theta^2 &= \frac{1}{6}F_{qq}(\omega^2 + \gamma\omega^1),\\
\theta^3 &= \frac{1}{36\alpha}F_{qq}\left(\omega^3 + \left(\gamma - \frac{1}{3}\right)F_q\omega^2 + \left(\frac{1}{2}\gamma^2 + K\right)\omega^1\right),\\
\theta^4 &= \frac{6\alpha}{F_{qq}}\omega^4,\\
\Omega^1 &= \frac{1}{F_{qq}}\left(-F_{qqq}\gamma^2 + \left(\frac{2}{3}F_{qqq}F_q + \frac{1}{3}F_{qq}^2 + 2F_{qqp}\right)\gamma + F_{qq}K_q\right.\\
&\quad \left. + 2F_{qqq}K - 2F_{qqy}\right)\omega^1 - \tfrac{\gamma}{\alpha}\,\mathrm{d}\alpha\\
\Omega^2 &= -\frac{1}{6\alpha}F_{qq}\left(\frac{1}{2}\gamma^2 + \frac{1}{3}F_q\gamma + K\right)\omega^4\\
&\quad + \frac{1}{6\alpha}\left(-\frac{1}{2}F_{qqq}\gamma^2 + \left(\frac{1}{3}F_{qqq}F_q + F_{qqp}\right)\gamma + F_{qqq}K - F_{qqy}\right)\omega^2\\
&\quad + \frac{1}{6\alpha}\left(-\frac{1}{2}F_{qqq}\gamma^3 + \left(\frac{1}{6}F_{qq}^2 + \frac{1}{3}F_{qqq}F_q + F_{qqp}\right)\gamma^2\right.\\
&\quad + (F_{qq}K_q - F_{qqy} + F_{qqq}K)\gamma - \frac{1}{3}F_{qq}F_{qy} - F_{qq}K_p - \frac{1}{3}F_{qq}F_qK_q\\
&\quad \left. + \frac{1}{3}F_{qq}^2K\right)\omega^1 + \frac{1}{6\alpha}F_{qq}\,\mathrm{d}\gamma,
\end{aligned} \tag{6}$$

where K denotes

$$K = \frac{1}{6}(F_{qx} + pF_{qy} + qF_{qp} + FF_{qq}) - \frac{1}{9}F_q^2 - \frac{1}{2}F_p$$

and ω^i, $i = 1, 2, 3, 4$ *are defined by the ODE via* (4).

Exterior derivatives of the above invariant forms read

$$\begin{aligned}
\mathrm{d}\theta^1 &= \Omega^1 \wedge \theta^1 + \theta^4 \wedge \theta^2,\\
\mathrm{d}\theta^2 &= \Omega^2 \wedge \theta^1 + a\theta^3 \wedge \theta^2 + b\theta^4 \wedge \theta^2 + \theta^4 \wedge \theta^3,\\
\mathrm{d}\theta^3 &= \Omega^2 \wedge \theta^2 - \Omega^1 \wedge \theta^3 + (2 - 2c)\theta^3 \wedge \theta^2 + e\theta^4 \wedge \theta^1 + 2b\theta^4 \wedge \theta^3,\\
\mathrm{d}\theta^4 &= \Omega^1 \wedge \theta^4 + f\theta^4 \wedge \theta^1 + (c - 2)\theta^4 \wedge \theta^2 + a\theta^4 \wedge \theta^3,\\
\mathrm{d}\Omega^1 &= (2c - 2)\Omega^2 \wedge \theta^1 - \Omega^2 \wedge \theta^4 + g\theta^1 \wedge \theta^2 + h\theta^1 \wedge \theta^3\\
&\quad + k\theta^1 \wedge \theta^4 - f\theta^2 \wedge \theta^4,\\
\mathrm{d}\Omega^2 &= \Omega^2 \wedge \Omega^1 - a\Omega^2 \wedge \theta^3 - b\Omega^2 \wedge \theta^4 + l\theta^1 \wedge \theta^2 + m\theta^1 \wedge \theta^3 + n\theta^1 \wedge \theta^4\\
&\quad + r\theta^2 \wedge \theta^3 + s\theta^2 \wedge \theta^4 - f\theta^3 \wedge \theta^4,
\end{aligned} \tag{7}$$

where $a, b, c, e, f, g, h, k, l, m, n, r, s$ are functions on $\mathcal{P}$, which can be simply calculated due to formulae (6). The simplest and the most symmetric case, when all the functions $a, b, c, e, f, g, h, k, l, m, n, r, s$ vanish, corresponds to the f.p. equivalence class of equation

$$y''' = \frac{3}{2}\frac{y''^2}{y'}.$$

In this case, the manifold $\mathcal{P}$ is (locally) the Lie group SO(2, 2) and the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2)$ is a basis of left invariant forms, which can be collected to the so(2, 2)-valued flat Cartan connection on $\mathcal{P} = \mathrm{SO}(2, 2)$. Since the Levi–Civita connection for the split signature metrics in four dimensions also takes value in so(2, 2), we ask under which conditions on f.p. equivalence classes of ODEs (1), Eqs. (7) may be interpreted as the structure equations for the Levi–Civita connection of a certain four-dimensional split signature metric G.

3. The construction of the metrics

It is convenient to change the basis of 1-forms $\theta^1, \theta^2, \theta^3, \theta^4, \Omega^1, \Omega^2$ on $\mathcal{P}$ to

$$\tau^1 = 2\theta^1 + \theta^4, \qquad \tau^2 = \Omega^2, \qquad \tau^3 = \Omega^2 + 2\theta^3, \qquad \tau^4 = \theta^4,$$
$$\Gamma_1 = \Omega^1, \qquad \Gamma_2 = \Omega^1 + 2\theta^2. \tag{8}$$

After this change, Eqs. (7) yield the formulae for the exterior differentials of $\tau^1, \tau^2, \tau^3, \tau^4, \Gamma_1, \Gamma_2$. These are the formulae (23) of Appendix A. They can be used to analyze the properties of the following bilinear tensor field

$$\tilde{G} = \tilde{G}_{ij}\tau^i\tau^j = 2\tau^1\tau^2 + 2\tau^3\tau^4 \tag{9}$$

on $\mathcal{P}$. The first question we ask here is the following: under which conditions on $a, b, c, e, f, g, h, k, l, m, n, r, s$ the first four of Eqs. (23) may be identified with

$$\mathrm{d}\tau^i + \Gamma^i{}_j \wedge \tau^j = 0,$$

where the 1-forms Γ^i_j, $i, j = 1, 2, 3, 4$ satisfy

$$\Gamma_{(ij)} = 0, \quad \text{and} \quad \Gamma_{ij} = \tilde{G}_{ik}\Gamma^k_j.$$

This happens if and only if

$$c = 0, \qquad l = 0, \qquad r = 0, \qquad s = 0. \tag{10}$$

Now, we call 1-forms Γ_1, Γ_2 as *vertical* and 1-forms $\tau^1, \tau^2, \tau^3, \tau^4$ as *horizontal*. To be able to interprete

$$R^i_j = \mathrm{d}\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j$$

as a curvature, we have to require that it is horizontal, i.e. contains no Γ_1, Γ_2 terms. This is equivalent to

$$m = 0, \qquad a = 0, \qquad g = 0, \qquad f = -b. \tag{11}$$

If these conditions are satisfied then the exterior derivatives of (23) give also

$$b = 0, \qquad h = 0. \tag{12}$$

Concluding, having conditions (10)–(12) satisfied, we have the following differentials of the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Gamma_1, \Gamma_2)$:

$$\begin{aligned}
d\tau^1 &= \Gamma_1 \wedge \tau^1,\\
d\tau^2 &= -\Gamma_1 \wedge \tau^2 + \tfrac{1}{2}n\tau^1 \wedge \tau^4,\\
d\tau^3 &= -\Gamma_2 \wedge \tau^3 + \left(\tfrac{1}{2}n - e\right)\tau^1 \wedge \tau^4,\\
d\tau^4 &= \Gamma_2 \wedge \tau^4,\\
d\Gamma_1 &= \tau^1 \wedge \tau^2 + \tfrac{1}{2}k\tau^1 \wedge \tau^4,\\
d\Gamma_2 &= \tfrac{1}{2}k\tau^1 \wedge \tau^4 - \tau^3 \wedge \tau^4,
\end{aligned} \tag{13}$$

and the following formulae for the matrix of 1-forms

$$\Gamma^i_{\ j} = \begin{pmatrix} -\Gamma_1 & 0 & 0 & 0\\ 0 & \Gamma_1 & 0 & -\frac{1}{2}n\tau^1 + (e - \frac{1}{2}n)\tau^4\\ \frac{1}{2}n\tau^1 - (e - \frac{1}{2}n)\tau^4 & 0 & \Gamma_2 & 0\\ 0 & 0 & 0 & -\Gamma_2 \end{pmatrix}.$$

Moreover, introducing the frame of the vector fields $(X_1, X_2, X_3, X_4, Y_1, Y_2)$ dual to the coframe $\tau^1, \ldots, \tau^4, \Gamma_1, \Gamma_2$ we get the following non-vanishing 2-forms $R^i_{\ j}$:

$$\begin{aligned}
R^1_{\ 1} &= -\tau^1 \wedge \tau^2 - \tfrac{1}{2}k\tau^1 \wedge \tau^4,\\
R^2_{\ 2} &= \tau^1 \wedge \tau^2 + \tfrac{1}{2}k\tau^1 \wedge \tau^4,\\
R^2_{\ 4} &= \tfrac{1}{2}k\tau^1 \wedge \tau^2 + \left(\tfrac{1}{2}n_4 + e_1 - \tfrac{1}{2}n_1\right)\tau^1 \wedge \tau^4 - \tfrac{1}{2}k\tau^3 \wedge \tau^4,\\
R^3_{\ 1} &= -\tfrac{1}{2}k\tau^1 \wedge \tau^2 - \left(\tfrac{1}{2}n_4 + e_1 - \tfrac{1}{2}n_1\right)\tau^1 \wedge \tau^4 + \tfrac{1}{2}k\tau^3 \wedge \tau^4,\\
R^3_{\ 3} &= \tfrac{1}{2}k\tau^1 \wedge \tau^4 - \tau^3 \wedge \tau^4,\\
R^4_{\ 4} &= -\tfrac{1}{2}k\tau^1 \wedge \tau^4 + \tau^3 \wedge \tau^4.
\end{aligned}$$

Here f_i denotes $X_i(f)$. It further follows that $Ric_{ij} = R^k_{\ ikj}$ satisfies

$$Ric_{ij} = -\tilde{G}_{ij}. \tag{14}$$

These preparatory steps enable us to associate with each f.p. equivalence class of ODEs (1) satisfying conditions (10)–(12) a four-manifold $\mathcal{M}$ equipped with a split signature Einstein metric G. This is done as follows.

- The system (13) guarantees that the distribution $\mathcal{V}$ spanned by the vector fields Y_1, Y_2 is integrable. The leaf space of this foliation is four-dimensional and may be identified with $\mathcal{M}$. We also have the projection $\pi : \mathcal{P} \to \mathcal{M}$.
- The tensor field $\tilde{G}$ is degenerate, $\tilde{G}(Y_1, \cdot) = 0$, $\tilde{G}(Y_2, \cdot) = 0$, along the leaves of $\mathcal{V}$. Moreover, equations (13) imply that

$$L_{Y_1}\tilde{G} = 0, \qquad L_{Y_2}\tilde{G} = 0.$$

  Thus, $\tilde{G}$ projects to a well-defined split signature metric G on $\mathcal{M}$.
- The Levi–Civita connection 1-form for G and the curvature 2-form, pull-backed via π^* to $\mathcal{P}$, identify with Γ^i_j and $R^i_{\ j}$, respectively.
- Thus, due to equations (14), the metric G satisfies the Einstein field equations with cosmological constant $\Lambda = -1$.

Below we find all functions $F = F(x, y, p, q)$ which solve conditions (10)–(12). This will enable us to write down the explicit formulae for the Einstein metrics G associated with the corresponding equations $y''' = F(x, y, y', y'')$.

The conditions $b = 0$, $c = 0$ in coordinates $x, y, p, q, \alpha, \gamma$ read

$$F_{qp} + \frac{1}{3}F_{qq} + 3K_q = 0, \qquad F_{qqq}\gamma - F_{qqp} - \frac{1}{3}F_{qqq}F_q + \frac{1}{6}F_{qq}^2 = 0.$$

The most general funtion $F(x, y, p, q)$ defining third-order ODEs satisfying these constraints is

$$F = \frac{3}{2}\frac{q^2}{p + \sigma(x, y)} + 3\frac{\sigma_x(x, y) + p\sigma_y(x, y)}{p + \sigma(x, y)}q + \xi(x, y, p),$$

where σ, ξ are arbitrary functions of two and three varaibles, respectively. Since the equations are considered modulo fiber preserving transformations, we can put $\sigma = 0$ by transformation $\overline{x} = x$ and $\overline{y} = \overline{y}(x, y)$ such that $\overline{y}_x = -\sigma(x, \overline{y}(x, y))$. Condition $l = 0$ now becomes

$$p^3\xi_{ppp} - 3p^2\xi_{pp} + 6p\xi_p - 6\xi = 0,$$

with the following general solution

$$\xi = A(x, y)p^3 + C(x, y)p^2 + B(x, y)p.$$

Hence F is given by

$$F = \frac{3}{2}\frac{q^2}{p} + A(x, y)p^3 + C(x, y)p^2 + B(x, y)p. \tag{15}$$

It further follows that it fulfills the remaining conditions $a = f = g = h = m = r = s = 0$ and that

$$k = -\frac{C}{4\alpha^2 p}, \qquad n = \frac{C_y - zC - 2A_x}{8\alpha^3 p}, \qquad e = \frac{1}{2}n + \frac{tC + 2B_y - C_x}{16\alpha^3 p^2}. \tag{16}$$

A straightforward application of Theorem 2.1 leads to the following expressions for the 'null coframe' $(\tau^1, \tau^2, \tau^3, \tau^4)$:

$$\begin{aligned}
\tau^1 &= 2\alpha\, \mathrm{d}y\\
\tau^2 &= (4\alpha)^{-1}[C\, \mathrm{d}x + (2A - z^2)\, \mathrm{d}y + 2\, \mathrm{d}z]\\
\tau^3 &= (4\alpha p)^{-1}[-(t + 2B)\, \mathrm{d}x - C\, \mathrm{d}y + 2\, \mathrm{d}t]\\
\tau^4 &= 2\alpha p\, \mathrm{d}x,
\end{aligned}$$

where the new coordinates z and t are

$$z = \frac{\gamma}{p}, \qquad t = \frac{q}{p} + \gamma.$$

This brings

$$\tilde{G} = 2(\tau^1\tau^2 + \tau^3\tau^4)$$

on $\mathcal{P}$ to the form that depends only on coordinates (x, y, z, t). Thus, $\tilde{G}$ projects to a well-defined split signature metric

$$G = -[t^2 + 2B(x, y)]\, \mathrm{d}x^2 + 2\, \mathrm{d}t\, \mathrm{d}x + [2A(x, y) - z^2]\, \mathrm{d}y^2 + 2\, \mathrm{d}z\, \mathrm{d}y$$

on a four-manifold $\mathcal{M}$ parmetrized by (x, y, z, t).

It follows from the construction that metric G is f.p. invariant. However, it does not yield all the f.p. information about the corresponding ODE. It is clear, since the function C which is proportional to the f. p. Cartan's invariant k of (13), is not appearing in the metric G. From the point of view of the metric, function C represents a 'null rotation' of coframe (τ^i). Thus it is not a geometric quantity. Therefore G, although f.p. invariant, can not distinguish between various f.p. nonequivalent classes of equations such as, for example, those with $C \equiv 0$ and $C \neq 0$. To fully distinguish all non-equivalent ODEs with (15) one needs additional structure than the metric G. This structure is only fully described by the bundle $\pi : \mathcal{P} \to \mathcal{M}$ together with the coframe $(\tau^1, \tau^2, \tau^3, \tau^4, \Gamma_1, \Gamma_2)$ of (13) on $\mathcal{P}$. An alternative description, more in the spirit of the split signature metric G, is presented in Section 5.

Now, Eqs. (14) imply that the metric G is Einstein with cosmological constant $\Lambda = -1$. The anti-selfdual part of its Weyl tensor is always of Petrov–Penrose type D. The selfdual Weyl tensor is of type II, if the functions A and B are generic. If $A = A(y)$ and $B = B(x)$ the selfdual Weyl tensor degenerates to a tensor of type D. Summing up we have following theorem.

Theorem 3.1. *Third-order ODE*

$$y''' = \frac{3}{2}\frac{y''^2}{y'} + A(x, y)y'^3 + C(x, y)y'^2 + B(x, y)y'$$

defines, by virtue of Cartan's equivalence method, a four-dimensional split signature metric

$$G = -[t^2 + 2B(x, y)]\,\mathrm{d}x^2 + 2\,\mathrm{d}t\,\mathrm{d}x + [2A(x, y) - z^2]\,\mathrm{d}y^2 + 2\,\mathrm{d}z\,\mathrm{d}y$$

which is Einstein

$$Ric(G) = -G$$

and has Weyl tensor $W = W^{\mathrm{ASD}} + W^{\mathrm{SD}}$ of Petrov type $D + II$, with the exception of the case $A = A(y)$, $B = B(x)$, when it is of type $D + D$. The metric G is invariant with respect to f.p. transformations of the variables of the ODE.

4. Uniqueness of the metrics

In this section we prove the following theorem.

Theorem 4.1. *The metrics of* Theorem 3.1 *are the unique family of metrics G, which are defined by f.p. equivalence classes of third-order ODEs and satisfy the following three conditions.*

- *The metrics are split signature, Einstein: $Ric(G) = -G$, and each of them is defined on four-dimensional manifold $\mathcal{M}$, which is the base of the fibration $\pi : \mathcal{P} \to \mathcal{M}$.*
- *The family contains a metric corresponding to equation $y''' = \frac{3}{2}\frac{y''^2}{y'}$.*
- *The tensor*

  $$\tilde{G} = \pi^* G = \mu_{ij}\theta^i\theta^j + \nu_{iA}\theta^i\Omega^A + \rho_{AB}\Omega^A\Omega^B,$$

  on $\mathcal{P}$, when expressed by the invariant coframe (θ^i, Ω^A) associated with the respective f.p. equivalence class, has the coefficients μ_{ij}, ν_{iA}, ρ_{AB}; $i, j = 1, \ldots, 4$; $A, B = 1, 2$ constant and the same for all the classes of the ODEs for which G is defined.

To prove the theorem, it is enough to show the uniqueness of G in the simplest case of equation $y''' = \frac{3}{2}\frac{y''^2}{y'}$, and to repeat the calculations of Section 3 for a generic equation. The following trivial proposition holds.

Proposition 4.2. *Let $\tilde{G}$ be a bilinear symmetric form of signature $(++--00)$ on $\mathcal{P}$, such that for a vector field N*

$$if \quad \tilde{G}(N, \cdot) = 0 \quad then \quad L_N\tilde{G} = 0. \tag{17}$$

A distribution spanned by such vector fields N is integrable and defines a four-dimensional manifold $\mathcal{M}$ as a space of its integral leaves. There exists exactly one bilinear form G on $\mathcal{M}$ with the property $\pi^* G = \tilde{G}$, where $\pi : \mathcal{P} \to \mathcal{M}$ is the canonical projection assigning a point of $\mathcal{M}$ to an integral leave of the distribution.

Our aim now is to find all the metrics $\tilde{G}$ of Proposition 4.2 which, when expressed by the coframe θ^i, Ω^A (or, equivalently, by τ^i, Γ_A), have constant coefficients. Let us consider the simplest case, corresponding to equation $y''' = \frac{3}{2}\frac{y''^2}{y'}$, for which all the invariant functions appearing in (7) and (23) vanish. $\mathcal{P}$ is now the Lie group SO(2, 2), $\tilde{G}$ is a form on its Lie algebra so(2, 2), the distribution spanned by the degenerate fields N is a two-dimensional subalgebra $\mathfrak{h} \subset$ so(2, 2). Finding $\tilde{G}$ is now a purely algebraic problem. In our case the basis (τ^i, Γ_A) satisfies

$$\begin{aligned} &\mathrm{d}\tau^1 = \Gamma_1 \wedge \tau^1, && \mathrm{d}\tau^3 = -\Gamma_2 \wedge \tau^3,\\ &\mathrm{d}\tau^2 = -\Gamma_1 \wedge \tau^2, && \mathrm{d}\tau^4 = \Gamma_2 \wedge \tau^4,\\ &\mathrm{d}\Gamma_1 = \tau^1 \wedge \tau^2, && \mathrm{d}\Gamma_2 = \tau^4 \wedge \tau^3, \end{aligned} \tag{18}$$

which agrees with a decomposition so(2, 2) = so(1, 2) ⊕ so(1, 2). A group of transformations preserving equations (18) is O(1, 2) × O(1, 2), that is the intersection of the orthogonal group O(2, 4) preserving the Killing form κ of so(2, 2) and the group GL(3) × GL(3) preserving the decomposition so(2, 2) = so(1, 2) ⊕ so(1, 2). Each coframe $(\tilde{\tau}^i, \tilde{\Gamma}_A)$, satisfying (18) is obtained by a linear transformation:

$$\begin{pmatrix} \tilde{\tau}^1\\ \tilde{\tau}^2\\ \tilde{\Gamma}_1 \end{pmatrix} = A \begin{pmatrix} \tau^1\\ \tau^2\\ \Gamma_1 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{\tau}^3\\ \tilde{\tau}^4\\ \tilde{\Gamma}_2 \end{pmatrix} = B \begin{pmatrix} \tau^3\\ \tau^4\\ \Gamma_2 \end{pmatrix}, \qquad A, B \in \mathrm{O}(1,2). \tag{19}$$

We use transformations (19) to obtain the most convenient form of the basis (N_1, N_2) of the subalgebra $\mathfrak{h} \subset$ so(2, 2). We write down the metric $\tilde{G}$ in the corresponding coframe $(\tilde{\tau}^1, \tilde{\tau}^2, \tilde{\tau}^3, \tilde{\tau}^4, \tilde{\Gamma}_1, \tilde{\Gamma}_2)$ and impose conditions (17). This conditions imply that the most general form of the metric is $\tilde{G} = 2u\tilde{\tau}^1\tilde{\tau}^2 + 2v\tilde{\tau}^3\tilde{\tau}^4$, where u, v are two real parameters. In such case, $[N_1, N_2] = 0$ and $\kappa(N_1, N_1) < 0$, $\kappa(N_2, N_2) < 0$. When written in terms of the coframe (τ^i, Γ_A), $\tilde{G}$ involves six real parameters $u, v, \mu, \phi, \nu, \psi$, however it appears, that only parameters u and v are essential; different choices of μ, ϕ, ν, ψ define different degenerate distributions spanned by N_1, N_2 and hence spaces $\mathcal{M}$ are different, but metrics G on them are isometric. Thus we can choose $\tilde{G} = 2u\tau^1\tau^2 + 2v\tau^3\tau^4$. Computing $\tilde{G}$ for $F = \frac{3}{2}\frac{q^2}{p}$, we have, in a suitable coordinate system (x, y, z, t),

$$G = -v[t^2 + 2B(x, y)]\,\mathrm{d}x^2 + 2v\,\mathrm{d}t\,\mathrm{d}x + u[2A(x, y) - z^2]\,\mathrm{d}y^2 + 2u\,\mathrm{d}z\,\mathrm{d}y.$$

Parameters u, v can be also fixed, if we demand G to be Einstein with cosmological constant $\Lambda = -1$. This is only possible if $u = 1$, $v = 1$. The tensor field $\tilde{G}$ defined in this way is unique and has the form

$$\tilde{G} = 2\tau^1\tau^2 + 2\tau^3\tau^4 = 2\Omega^2(2\theta^1 + \theta^4) + 2\theta^4(2\theta^3 + \Omega^2).$$

This formula is used in the generic case explaining our choice of the coframe (8) and the metric (9). This finishes the proof of Theorem 4.1.

5. The Cartan connection and the distinguished class of ODEs

Here we provide an alternative description of the f.p. equivalence class of third-order ODEs corresponding to $F = F(x, y, p, q)$ of (15). We consider a four-dimensional manifold $\mathcal{M}$ parametrized by (x, y, z, t). Then the geometry of a f.p. equivalence class of ODEs (15) is in one to one correspondence with the geometry of a class of coframes

$$\begin{aligned} \tau_0^1 &= \mathrm{d}y \\ \tau_0^2 &= \tfrac{1}{2}[C\,\mathrm{d}x + (2A - z^2)\,\mathrm{d}y + 2\,\mathrm{d}z] \\ \tau_0^3 &= \tfrac{1}{2}[-(t + 2B)\,\mathrm{d}x - C\,\mathrm{d}y + 2\,\mathrm{d}t] \\ \tau_0^4 &= \mathrm{d}x, \end{aligned} \tag{20}$$

on $\mathcal{M}$ given modulo a special SO(2, 2) transformation

$$\tau_0^i \mapsto \tau^i = h^i{}_j \tau_0^j, \quad \text{where} \quad (h^i{}_j) = \begin{pmatrix} 2\alpha & 0 & 0 & 0 \\ 0 & (2\alpha)^{-1} & 0 & 0 \\ 0 & 0 & (2\alpha p)^{-1} & 0 \\ 0 & 0 & 0 & 2\alpha p \end{pmatrix}. \tag{21}$$

The Cartan equivalence method applied to the question if two coframes (20) are transformable to each other via (21) gives the full system of invariants of this geometry. These invariants consist of (i) a fibration $\pi : \mathcal{P} \to \mathcal{M}$ of Section 3, which now becomes a Cartan bundle $\mathcal{H} \to \mathcal{P} \to \mathcal{M}$ with the two-dimensional structure group $\mathcal{H}$ generated by $h^i{}_j$, and (ii) of an so(2, 2)-valued Cartan connection ω described by the coframe $(\tau^1, \tau^2, \tau^3, \tau^4, \Gamma_1, \Gamma_2)$ of (13) on $\mathcal{P}$. Explicitely, the connection ω is given by

$$\omega^i{}_j = \begin{pmatrix} -\frac{1}{2}(\Gamma_1 + \Gamma_2 + \tau^4) & 0 & \tau^1 & -\frac{1}{2}\tau^4 \\ 0 & \frac{1}{2}(\Gamma_1 + \Gamma_2 + \tau^4) & -\Gamma_2 + \tau^3 - \frac{1}{2}\tau^4 & -\frac{1}{2}\tau^2 \\ \frac{1}{2}\tau^2 & \frac{1}{2}\tau^4 & \frac{1}{2}(\Gamma_1 - \Gamma_2 - \tau^4) & 0 \\ \Gamma_2 - \tau^3 + \frac{1}{2}\tau^4 & -\tau^1 & 0 & \frac{1}{2}(-\Gamma_1 + \Gamma_2 + \tau^4) \end{pmatrix}.$$

To see that this is an so(2, 2) connection it is enough to note that $g_{ij}\omega^j{}_k + g_{kj}\omega^k{}_i = 0$ with the matrix g_{ij} given by

$$g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now, Eqs (13) are interpreted as the requirement that the curvature

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega$$

of this connection ω has a very simple form

$$\Omega = \begin{pmatrix} -\frac{1}{2}k & 0 & 0 & 0 \\ 0 & \frac{1}{2}k & \frac{1}{2}(-k+n-2e) & -\frac{1}{4}n \\ \frac{1}{4}n & 0 & 0 & 0 \\ \frac{1}{2}(k-n+2e) & 0 & 0 & 0 \end{pmatrix} \tau^1 \wedge \tau^4,$$

where n, e and k are given by (16). The connection ω and its curvature Ω yields all the f.p. information of the equation corresponding to (15). In particular, all the equations with $k = n = e = 0$ are f.p. equivalent, all having the vanishing curvature of their Cartan connection ω.

It is interesting to search for a split signature 4-metric H for which the connection ω is the Levi–Civita connection. The general form of such metric is

$$H = g_{ij} T^i T^j,$$

where (T^1, T^2, T^3, T^4) are four linearly independent 1-forms on $\mathcal{P}$ which staisfy

$$\mathrm{d}T^i + \omega^i_{\ j} \wedge T^j = 0. \tag{22}$$

Thus, for such H to exist, the 1-forms (T^1, T^2, T^3, T^4) must also satisfy the integrability conditions of (22),

$$\Omega^i_{\ j} \wedge T^j = 0,$$

which are just the Bianchi identities for ω to be the Levi–Civita connection of metric H. These identities provide severe algebraic constraints on the possible solutions (T^i). Using them, under the assumption that $C(x, y) \neq 0$ in the considered region of $\mathcal{P}$, we found all (T^i)s satisfying (22). Thus, with every triple $C \neq 0, A, B$ corresponding to an ODE given by F of (15), we were able to find a split signature metric H for which connection ω is the Levi–Civita connection. Surprisingly, given A, B and $C \neq 0$ the general solution for (T^i) involves four *free* real functions. Two of these functions depend on six variables and the other two depend on two variables. Thus, each f.p. equivalence class of ODEs representd by F of (15) defines a large family of split signature metrics H for which ω is the Levi–Civita connection.[1] Writing down the explicit formulae for these metrics is easy, but we do not present them here, due to their ugliness and due to the fact that, regardless of the choice of the four free functions, they never satisfy the Einstein equations. The proof of this last fact is based on lengthy calculations using the explicit forms of the general solutions for (T^i).

[1] The four-manifold on which each of these metrics resides is the leaf space of the two-dimensional integrable distribution on $\mathcal{P}$ which anihilates forms (T^1, T^2, T^3, T^4).

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Appendix A

In this appendix we give the formulae for the differentials of the transformed Cartan invariant coframe $(\tau^1, \tau^2, \tau^3, \tau^4, \Gamma_1, \Gamma_2)$ on $\mathcal{P}$. These are:

$$\mathrm{d}\tau^1 = \Gamma_1 \wedge \tau^1 + \frac{1}{2}c\Gamma_1 \wedge \tau^4 - \frac{1}{2}c\Gamma_2 \wedge \tau^4 + \frac{1}{2}f\tau^4 \wedge \tau^1 - \frac{1}{2}a\tau^4 \wedge \tau^2 + \frac{1}{2}a\tau^4 \wedge \tau^3, \tag{23a}$$

$$\begin{aligned}\mathrm{d}\tau^2 = {} & \frac{1}{4}l\Gamma_1 \wedge \tau^1 + \left(\frac{1}{4}r - 1\right)\Gamma_1 \wedge \tau^2 - \frac{1}{4}r\Gamma_1 \wedge \tau^3 - \left(\frac{1}{4}l + \frac{1}{2}s\right)\Gamma_1 \wedge \tau^4 \\ & - \frac{1}{4}l\Gamma_2 \wedge \tau^1 - \frac{1}{4}r\Gamma_2 \wedge \tau^2 + \frac{1}{4}r\Gamma_2 \wedge \tau^3 + \left(\frac{1}{4}l + \frac{1}{2}s\right)\Gamma_2 \wedge \tau^4 \\ & + \frac{1}{4}m\tau^2 \wedge \tau^1 - \frac{1}{4}m\tau^3 \wedge \tau^1 - \frac{1}{2}n\tau^4 \wedge \tau^1 + \frac{1}{2}a\tau^3 \wedge \tau^2 \\ & + \left(\frac{1}{4}m - \frac{1}{2}f + b\right)\tau^4 \wedge \tau^2 + \left(\frac{1}{2}f - \frac{1}{4}m\right)\tau^4 \wedge \tau^3,\end{aligned} \tag{23b}$$

$$\begin{aligned}\mathrm{d}\tau^3 = {} & \frac{1}{4}l\Gamma_1 \wedge \tau^1 + \left(c + \frac{1}{4}r\right)\Gamma_1 \wedge \tau^2 - \left(c + \frac{1}{4}r\right)\Gamma_1 \wedge \tau^3 \\ & - \left(\frac{1}{4}l + \frac{1}{2}s\right)\Gamma_1 \wedge \tau^4 + \frac{1}{4}l\Gamma_2 \wedge \tau^1 - \left(c + \frac{1}{4}r\right)\Gamma_2 \wedge \tau^2 \\ & + \left(c + \frac{1}{4}r - 1\right)\Gamma_2 \wedge \tau^3 + \left(\frac{1}{4}l + \frac{1}{2}s\right)\Gamma_2 \wedge \tau^4 + \frac{1}{4}m\tau^2 \wedge \tau^1 \\ & - \frac{1}{4}m\tau^3 \wedge \tau^1 + \left(e - \frac{1}{2}n\right)\tau^4 \wedge \tau^1 + \frac{1}{2}a\tau^3 \wedge \tau^2 \\ & + \left(\frac{1}{4}m - b - \frac{1}{2}f\right)\tau^4 \wedge \tau^2 + \left(2b + \frac{1}{2}f - \frac{1}{4}m\right)\tau^4 \wedge \tau^3,\end{aligned} \tag{23c}$$

$$\mathrm{d}\tau^4 = +\frac{1}{2}c\Gamma_1 \wedge \tau^4 + \left(1 - \frac{1}{2}c\right)\Gamma_2 \wedge \tau^4 + \frac{1}{2}f\tau^4 \wedge \tau^1 - \frac{1}{2}a\tau^4 \wedge \tau^2 + \frac{1}{2}a\tau^4 \wedge \tau^3, \tag{23d}$$

$$\begin{aligned} \mathrm{d}\Gamma_1 = {} & \frac{1}{4}g\Gamma_1\wedge\tau^1 + \left(\frac{1}{2}f-\frac{1}{4}g\right)\Gamma_1\wedge\tau^4 - \frac{1}{4}g\Gamma_2\wedge\tau^1 + \left(\frac{1}{4}g-\frac{1}{2}f\right)\Gamma_2\wedge\tau^4 \\ & + \left(\frac{1}{4}h+c-1\right)\tau^2\wedge\tau^1 + -\frac{1}{4}h\tau^3\wedge\tau^1 - \frac{1}{2}k\tau^4\wedge\tau^1 \\ & + \left(\frac{1}{4}h+c\right)\tau^4\wedge\tau^2 - \frac{1}{4}h\tau^4\wedge\tau^3, \end{aligned} \tag{23e}$$

$$\begin{aligned} \mathrm{d}\Gamma_2 = {} & \frac{1}{4}g\Gamma_1\wedge\tau^1 - \frac{1}{2}a\Gamma_1\wedge\tau^2 + \frac{1}{2}a\Gamma_1\wedge\tau^3 + \left(b+\frac{1}{2}f-\frac{1}{4}g\right)\Gamma_1\wedge\tau^4 \\ & - \frac{1}{4}g\Gamma_2\wedge\tau^1 + \frac{1}{2}a\Gamma_2\wedge\tau^2 - \frac{1}{2}a\Gamma_2\wedge\tau^3 + \left(\frac{1}{4}g-b-\frac{1}{2}f\right)\Gamma_2\wedge\tau^4 \\ & + \left(\frac{1}{4}h+c\right)\tau^2\wedge\tau^1 - \frac{1}{4}h\tau^3\wedge\tau^1 - \frac{1}{2}k\tau^4\wedge\tau^1 + \left(\frac{1}{4}h+c\right)\tau^4\wedge\tau^2 \\ & + \left(1-\frac{1}{4}h\right)\tau^4\wedge\tau^3. \end{aligned} \tag{23f}$$

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